

FOUNDATIONS OF GEOMETRIC COGNITION

Mateusz Hohol

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The cognitive foundations of geometry have puzzled academics for a long time, and even today are mostly unknown to many scholars, including mathematical cognition researchers.

Foundations of Geometric Cognition shows that basic geometric skills are deeply hardwired in the visuospatial cognitive capacities of our brains, namely spatial navigation and object recognition. These capacities, shared with nonhuman animals and appearing in early stages of human ontogeny, cannot, however, fully explain a uniquely human form of geometric cognition. In the book, Hohol argues that Euclidean geometry would not be possible without the human capacity to create and use abstract concepts, demonstrating how language and diagrams provide cognitive scaffolding for abstract geometric thinking, within a context of a Euclidean system of thought.

Taking an interdisciplinary approach and drawing on research from diverse fields, including psychology, cognitive science, and mathematics, this book is a must-read for cognitive psychologists and cognitive scientists of mathematics, alongside anyone interested in mathematical education or the philosophical and historical aspects of geometry.

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CONTENTS

<i>Preface</i>	<i>ix</i>
<i>Acknowledgments</i>	<i>xv</i>
1 Geometric thinking, the paradise of abstraction	1
1.1 Introduction and synopsis of the chapter	1
1.2 The geometric roots of mathematical thinking	2
1.3 Geometric intuition as a philosophical locus classicus	10
1.4 The development of geometric skills as a psychological problem	17
1.5 Euclid in the classroom	25
1.6 How cognitive science discovered and forgot Euclid	31
1.7 Summary	40
Notes	41
2 The hardwired foundations of geometric cognition	45
2.1 Introduction and synopsis of the chapter	45
2.2 Hardwired sensitivity to geometry	46
2.3 In search of a theoretical framework	53
2.4 Causal factors: Core systems of geometry	62
2.5 The evolutionary adaptiveness of core geometry	66
2.6 Phylogeny of core geometry	68
2.7 Ontogeny: Toward a new representational system	72
2.8 Summary	76
Notes	78

3	Embodiment and abstraction	81
3.1	Introduction and synopsis of the chapter	81
3.2	The classic cognitive science of concepts	84
3.3	The embodiment and its challenges	89
3.4	Embodied theories of abstract concepts	97
3.5	Are abstract concepts really embodied?	103
3.6	Into the realm of abstraction: Through the body and beyond	108
3.7	Summary	115
	Notes	117
4	Cognitive artifacts and Euclid: Diagrams and formulae	121
4.1	Introduction and synopsis of the chapter	121
4.2	The lettered diagram as a cognitive artifact	124
4.3	The professional language of geometry: Another cognitive artifact	130
4.4	Where do the necessity and generality of Euclid's proofs come from?	134
4.5	Summary	139
	Notes	140
	Conclusions and future directions for research	143
	<i>References</i>	147
	<i>Index</i>	175

PREFACE

The prominent philosopher, logician, and mathematician Bertrand Russell confessed in his *Autobiography* that:

At the age of eleven, I began Euclid, with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined that there was anything so delicious in the world. After I had learned the fifth proposition, my brother told me that it was generally considered difficult, but I had found no difficulty whatever. This was the first time it had dawned upon me that I might have some intelligence. (Russell, 2009, p. 25)

I do not know of many people (in fact, not even one) who could honestly say something like Russell about their first encounter with geometry. Learning geometry from Euclid's original masterpiece, *Elements*, is not that common, either, even at the higher stages of one's formal education. We also do not encounter Euclid's famous fifth postulate, the one that so fascinated the 11-year-old Russell, unless we delve deeper into mathematics. Yet, each of us has encountered Euclidean plane geometry, enriched with some historically more recent inventions, such as a Cartesian coordinate system, at the very earliest stages of our schooling.

Besides arithmetic, most of the contemporary students and laymen alike perceive Euclidean geometry as the prototypical subject of mathematical education. Learning the principles of geometry, in a similar manner to numerical knowledge, plays a pivotal role in the acquisition of mathematical competencies that are useful in everyday life. Euclidean geometry is also extremely significant from the perspective of the history of mathematics. Hellenistic mathematics, including number theory, emerged from the use of geometric concepts and methods. Delving deeper still, the axiomatic-deductive approach to geometry developed by Greeks

and depicted in Euclid's masterpiece established a rigorous pattern of the philosophical discourse, or the rational thinking in general, for many centuries. Moreover, many prominent scientists admit that geometric imagination plays a great role in their mathematical thinking. Sir Roger Penrose (2004, 2018), for instance, used to say that he like considering problems geometrically. The same is true of Richard Feynman, as neatly described in his biography by Gleick (2011):

In high school he had not solved Euclidean geometry problems by tracking proofs through a logical sequence, step by step. He had manipulated the diagrams in his mind: he anchored some points and let others float, imagined some lines as stiff rods and others as stretchable bands, and let the shapes slide until he could see what the result must be. These mental constructs flowed more freely than any real apparatus could. Now, having assimilated a corpus of physical knowledge and mathematical technique, Feynman worked the same way. The lines and vertices floating in the space of his mind now stood for complex symbols and operators. They had a recursive depth; he could focus on them and expand them into more complex expressions, made up of more complex expressions still. He could slide them and rearrange them, anchor fixed points and stretch the space in which they were embedded. Some mental operations required shifts in the frame of reference, reorientations in space and time. The perspective would change from motionlessness to steady motion to acceleration. (p. 161)

These insights correspond with the frequently cited, and at the same time highly controversial, observation of Henri Poincaré (1929) that mathematicians can be divided into two camps: analysts and geometers, wherein it “does not prevent the one sort from remaining analysts even when they work at geometry, while the others are still geometers even when they occupy themselves with pure analysis” (p. 210; see Hadamard, 1945). All of this makes geometric thinking a fascinating topic.

Like any human intellectual enterprise, Euclidean geometry also emerges from cognitive processes and the activity of our brains. Nevertheless, the cognitive origins of geometry remained puzzling for a long time. A cognitive revolution that occurred in the mid-1950s, together with the later development of a new discipline called cognitive science, made this subject explorable in a scientific way. Nevertheless, the foundations of geometric cognition remain mostly unknown even today for the majority of mathematicians, historians of science, educational researchers, philosophers, psychologists, and cognitive scientists. Furthermore, even those of the latter who are interested in mathematical cognition focus primarily on the cognitive processing of numbers and calculations. This is manifested in the fact that the problem of geometry is essentially absent in most of the fundamental monographs in the field of mathematical cognition. We can observe a similar pattern in the cases of scientific conferences and journals. In contrast to the processing of numbers, there is no cyclical conference or peer-reviewed journal specializing in geometric cognition.

Although I have decided to limit my investigation in this book to Euclidean geometry, developed in ancient Greece but still taught to some degree to children today, starting in primary school, identifying crucial properties of geometric cognition remains a difficult task.¹ Despite the fact that children and the majority of educated adults are unfamiliar with proving theorems in an axiomatic-deductive fashion, leaving this method of reasoning aside entirely would make my investigation grossly incomplete. The proof is “a hard core” of Euclid’s contribution to the whole of mathematics. According to this fact, I try to explain not only where elementary manifestations of geometric cognition, such as sensitivity to angle, length (distance), and sense (left–right direction) come from, but also how the processing of abstract geometric concepts works and how Euclidean proofs that provide general results in a necessary way are cognitively possible at all. My proposal for sketching the account of the cognitive foundations of Euclidean geometry involves the following desiderata:²

(D1) The account should recognize whether the cognitive capacities that are necessary to engage with Euclidean geometry are “hardwired,” or whether they are rather constructed through individual learning.

(D2) The account should describe how these capacities are combined during ontogeny into a system of abstract geometric concepts.

(D3) The account should elucidate how the mind/brain of the human being (especially if it is constrained by the body and environment) is able to process abstract concepts at all.

(D4) The account should elucidate the geometric proof characterized by epistemic virtues: compelling power (or necessity) and generality of providing results.

Let me briefly explain how I intend to explore the above desiderata in the following chapters. In Chapter 1, I investigate different perspectives on geometric thinking, involving the history of mathematics, philosophy, early experimental psychology, education research, and, finally, interdisciplinary cognitive science, which will be further explored in subsequent chapters. This review chapter, which is the most extensive, will familiarize the reader with the crucial notions of Euclidean geometry and existing approaches to mathematical cognition, as well as the research problems that are particularly associated with the desiderata (D1–D4).

In Chapter 2, I attempt to identify the “hardwired” foundations of geometric cognition, namely those occurring in human beings at an early developmental stage and shared with nonhuman animals. Adopting Tinbergen’s strategy of explanatory questions I show that the sensitivity to elementary Euclidean properties is not a uniquely culture-dependent human skill that emerges when learning geometry in school. This sensitivity is observed in the context of the recognition of shapes and spatial navigation in many animal species and human infants in many cultures. Therefore, I defend a version of the “hardwiredness” of elementary geometric cognition (D1), elaborating upon this notion by grounding

it in the findings of various branches of cognitive science and related fields as cognitive and developmental psychology, neuroscience, evolutionary biology, ethology, comparative cognition, and behavioral robotics. Bearing in mind that only humans have developed, and are capable of assimilating, full-blooded Euclidean geometry, in this chapter I also investigate the limits of hardwired geometry and how children go beyond them. Thereby, I show that the process of the acquisition of the abstract conceptual structures of geometry is first mediated at the preschool level by enculturation with spatial language and map-like scale models (D2).

It is a truism to say that the concepts of Euclidean geometry are abstract in nature (D3). On the other hand, there is much evidence to show that the body and physical surroundings constrain the thinking of human beings. Therefore, in Chapter 3, I investigate how the processing of abstract concepts, something which seems to require us to reach beyond our proximal experience, is at all cognitively possible. I start by discussing the classic view of computational cognitive science on conceptual processing and show that it faced severe challenges (e.g., the symbol grounding problem) that stimulated cognitive science to shift the research paradigm. In this context I describe the emergence of embodied cognitive science while simultaneously claiming that in its strong version, where it assumes that the sensorimotor system of the brain both serves as the conceptual vehicle and determines the conceptual content, it is ill suited for the elucidation of geometric abstracts. I defend the claim that (D3) can be fulfilled by adopting the moderate version of embodiment, one that makes room for the shaping of the content of abstract concepts by internalizing natural language. Following Lev Vygotsky and contemporary theorists of the moderate version of embodied cognitive science, I show that, by virtue of its social nature and computational properties, human language serves as scaffolding for further learning. In other words, it is a cognitive artifact that makes establishing and using abstract concepts cognitively possible.

The above-summarized chapter does not directly answer the question of the origin of “the power of proof,” or the epistemic virtues of Euclidean geometry. Therefore, in Chapter 4, I seek a cognitive base for the compelling power (or necessity) of Euclidean reasonings and the generality of provided results (D4). To this end I shift the perspective of my investigation from experimentally oriented cognitive science to the cognitive history of geometry as developed by Reviel Netz. I make use of the notion of the cognitive artifact introduced in Chapter 3 and trace the role of two intertwined inventions of the ancient Greeks, namely lettered diagrams and well-regulated professional language, which helped to build a cognitive niche within which the necessity and generality of geometric proofs became possible. In this context, I hypothesize that the professional language of geometry—one that is mutually interconnected with diagrams through letters associated with geometric points—is characterized by the computational properties enhancing the hardwired cognitive capacities of the human being to a greater extent than ordinary or everyday language. At the end

of the book, I list the limits of my investigation and outline the perspectives for further research on geometric cognition.

Notes

1. Cognitive scientists sometimes do not perceive geometry as a phenomenon to be explained, or explanandum, but they use geometric or topological structures for modelling cognitive processes and representations. Geometry is used to explain mental phenomena, or plays a role in the explanans, for example, in Gärdenfors' (2004, 2014) theory of conceptual spaces.
2. Note that the following list is a modified version of one introduced previously in our article (Hohol & Miłkowski, 2019). The current proposal is undoubtedly not final, and does not pretend to be complete. Assuredly, the desiderata for the study of geometric cognition will change during the evolution of the cognitive science of mathematics and related fields.