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The Normativity of Mathematics. A Neurocognitive Approach[1]

Logic – it is an ethics of speech and thought
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1. Mathematics As a Normative Science

The term normativity is usually associated with ethics, law and language. In the case of ethics and law, this problem appears in many ways, most commonly in the context of the so-called naturalistic fallacy (the is-ought problem),[2] and in the case of language – in discussions concerning the normativity of me-

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[2] Cf. D. Hume, *A Treatise on Human Nature*, Dover Publications, New York 2003 (1739–40), p. 304; G.E. Moore, *Principia Ethica*, Cambridge University Press, Cambridge 1903, chapter 1, §10; J. Stelmach, *And if There is No 'Ought'?*, [in:] *Studies in the Philosophy of Law 6. The Normativity of Law*, eds. J. Stelmach, B. Brożek, Copernicus Center Press, Kraków 2011, pp. 15–20; J. Stelmach, *Naturalistic Fallacy and Antinaturalistic Fallacy in Normative Discourse*, this volume; A. Brożek, *Methodological Status of Naturalistic Fallacy*, this volume.

aning.[3] Much less frequently – although by no means never – we do speak about normativity in the context of the formal sciences, such as logic and mathematics. This essay is about the problem of the normativity of the latter disciplines, with particular emphasis on mathematics. One of the theses which I intend to defend is the statement that, despite the fact that in many domains such as language, morality and mathematics the term takes different semantic nuances, there is a common base for them, which I will call a proto-normativity or proto-rules. The main instrument which I intend to use is the embodied-embedded mind, one of the interpretative paradigms of cognitive neuroscience.

After an approximation of the problem in the first "negative" part of this paper, I will review the normativity in the context of the classical views in the philosophy of mathematics, such as Platonism and formalism. I will try to show, that Platonism and formalism do not provide an adequate solution to the problem of normativity. In the second "positive" part of the paper I will review the problem of normativity through the prism of the embodied-embedded mind paradigm. In this part I will give arguments supporting the thesis according to which, although we are just taking our first steps in the neurocognitive approach to mathematics, our hopes for a solution to the riddle of normativity should be associated with this approach.

What exactly is the normativity of mathematics? I will try to approximate this problem in two steps – firstly referring to the famous "Wittgenstein paradox" and, secondly, by showing, after Robert Hanna, three features of the intrinsic normativity of formal sciences. The famous part of paragraph 201 of the "Philosophical Investigations" reads as follows:

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[3] Cf. S. Kripke, *Wittgenstein on Rules and Private Language*, Harvard University Press, Cambridge 1982; K. Glüer, P. Pagin, *Rules of Meaning and Practical Reasoning*, "Synthese" 1999, no. 117, pp. 207–227; K. Glüer, Å. Wikforss, *Against Content Normativity*, "Mind" 2009, no. 118, pp. 31–70; B. Brożek, *The Normativity of Language*, this volume; A. Shaw, *The Perspectivist Account of the Normativity of Meaning Debate*, this volume.

> This was our paradox: no course of action could be determined by a rule, because every course of action can be made out to accord with the rule. The answer was: if everything can be made out to accord with the rule, then it can also be made out to conflict with it. And so there would be neither accord nor conflict here.[4]

The above paradox is one of the most important parts of Wittgenstein's work concerning rule-following. On of the most oft cited examples illustrating the problem of rule-following is the rule in mathematics which orders us to add 2 to any number ("+2"). Wittgenstein encourages us to imagine a young student of mathematics who, starting from zero, iteratively adds the number 2. As a result he gets the following numbers: 2, 4, 6, 8, 10, 12 etc. Addition proceeds – from our perspective – properly, until reaching 1000. Because the next following numbers are: 1004, 1008, 1012. Wittgenstein writes:

> We say to him: "Look what you've done!" – Pie doesn't understand. We say: "You were meant to add 2 look how you began the series!" – He answers: "Yes, isn't it right? I thought that was how I was meant to do it." – Or suppose he pointed to the series and said: "But I went on in the same way." – It would now be no use to say: "But can't you see....?" – and repeat the old examples and explanations. – In such a case we might say, perhaps: It comes natural to this person to understand our order with our explanations as we should understand the order: "Add 2 up to 1000, 4 up to 2000, 6 up to 3000 and so on."[5]

The example showed by Wittgenstein leads to a question – what exactly gives us certainty or correctness of the results achieved with the rule "+2"? How can we know whether we have followed this

[4] L. Wittgenstein, *Philosophical Investigations*, trans. G.E.M. Anscombe, Blackwell, Oxford 1986, §201.
[5] *Ibidem*, §185.

rule properly? How can we know whether this rule also applies in the case of great numbers, such as e.g. 20628344453451134534598? The problem becomes nontrivial when, instead of the rule "+2", we consider any different mathematical operation. How can we know whether the results achieved in advanced branches of mathematics, such as e.g. noncommutative geometry, are not only correct from the subjective point of view, but objective and necessary? Or, in other words, how can we know whether any rules applies in them?

Robert Hanna said that formal sciences, such as mathematics and logic, have an internal feature, which means that they are *intrinsically categorically normative*.[6] By his statement he meant that their rules are binding in any *situation,* in any *time* and for any *user*. Hanna distinguishes three features of the normativity of formal sciences.[7] The *First* of them says, that every formal system has two aspects: *factual*, which describes inter alia the relation of consequences and the *normative* aspect, which is corresponding with it. It is clearly visible e.g. in the natural deduction system, in whom inference rules are directly given, and in whom the logic truth is deducted from conditions. What's more, in his opinion:

> this can be seen in the fact that the protologic, as a set of logical principles and concepts for constructing logical systems, is inherently normative precisely insofar as it is a set of schematic per-missions to construct logical systems in just these ways and no others.[8]

Protologic doesn't say *how,* in practice, to construct formal systems. Hanna compares it to the Universal Grammar of Noam Chomsky, which is prescriptive, but not descriptive. Protologic sets the field of possibility, based on cognitive accessories, in which natural languages may be constructed. But it doesn't describe how those lan-

[6] Cf. R. Hanna, *Rationality of Logic*, The MIT Press, Cambridge-London 2006, p. 214.
[7] Cf. *ibidem*, p. 209.
[8] *Ibidem*.

guages are constructed in the real world. The second feature which Hanna distinguishes assumes that normative and innormative aspects of the formal systems are mutually complementary and are not reductive to each other. It can be said that, they are in state of 'nonlinear entanglement'. Finally, the third feature noticed by him consists in that proctologic is definitely normative in relation to human reasoning. In such an approach, the rules of protologic would be same as the basic rules of rationality that guide the subject.

2. The Normativity of Mathematics. Classical Philosophical Approaches

The normativity of Platonism

We may say that, statistically, the most common philosophical approach among the mathematicians is the Platonism.[9] When we are speaking about mathematical Platonism, we first think about it's supporters, such as Kurt Gödel, Roger Penrose or Michael Heller. Individual supporters of this philosophical approach, however, differ in case of particular resolutions, so it is difficult to formulate an unambiguous description of Platonism.[10] Instead of trying to construct an essential definition of this approach, it is better to review it on the basis of 'family similarities'. For example, Kurt Gödel – one of the representative Platonists – understood mathematics in the following way:

> Classes and concepts may, however, also be conceived as real objects namely classes as "pluralities of things" or as structures consisting of a plurality of things and concepts as the properties and relations of things existing independently of our definitions and constructions. It

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[9] Cf. R. Hersh, *On Platonism*, "European Mathematical Society Newsletter" 2007, vol. 64, pp. 24–25.

[10] Cf. M. Balaguer, *Platonism and Anti-Platonism in Mathematics*, Oxford University Press, New York-Oxford 1998.

> seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence.[11]

Many mathematicians who admit to Platonism see in it, above all, the guarantee of stability and certainty of mathematical results. In this context, it is worth quoting words of Alain Connes – Fields medal winner and one of the creators of the noncommutative geometries, mentioned before:

> I think I'm fairly close to the realist point of view. The prime numbers, for example , which, as far as I'm concerned, constitute a more stable reality then the material reality that surrounds us. The working mathematician can be likened to an explorer who sets out to discover the world.[12]

In contrast to this stability is the variability of the physical world. That is the reason for tendency to call mathematical objects abstractive, which we should understand as beyond time and independent from the physical world. It should be noted that not all of the mathematicians who consider themselves as Platonists believe in the existence of the 'platonic universe of abstract objects. What may be considered strange is that the problem of the ontological status of the mathematical objects most often becomes of secondary importance compared to the conviction of necessity and immutability of the mathematical truth. In the times of active dispute about the foundations of mathematics, the Platonism was often (but obviously not always) connected by logicism – the view that sought the basis of the whole of mathematics in logic (the first-order logic). In such a case, the nature of the

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[11] K. Gödel, *Russell's Mathematical Logic*, [in:] *Philosophy of Mathematics. Selected Readings*, eds. P. Benacerraf, H. Putnam, Cambridge University Press, Cambridge 1983, p. 456.

[12] J.-P. Changeux, A. Connes, *Conversations on Mind, Matter and Mathematics*, trans. M.B. DeBovoise, Princeton University Press, Princeton-New Jersey 1995, p. 12.

normativity of mathematics and logics would be not only similar, but actually identical.

Alain Cannes, cited above, emphasizes one more relevant feature of the mathematics which is accepted by most of the Platonists:

> The truth of Euclid's theorem about prime numbers doesn't depend on such-and-such a mode of perception. While it's true that mathematicians is used as a language by the other sciences, reducing it to a mere language would be a serious mistake.[13]

What is more, according to most Platonists, mathematics does not only become reduced to the language, but is independent from human linguistic abilities. Interestingly, in this context, mathematical Platonists willingly refer to empirical data, such as described by Alison Gopnik and collaborators, on the genetic impairments of speech which are connected with the occurrence of above-avarage mathematical abilities of the affected people.[14] An example often shown in this context is Albert Einstein, diagnosed with dislalia in childhood, a disorder in acquiring the proper articulation of speech.

Mathematical Platonism is not only an ontological approach. It also affects the methodology of mathematics in a significant way. The most meaningful feature which distinguish mathematicians-Platonists from mathematicians-intuitionists is an acceptance of unconstructive evidence. It happens to be so because, according to the Platonists, in order to prove the existence of an object in the mathematical universe, it is enough to show it's consistency. Another important issue which most Platonists – but definitely not all of them – are interested in is *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*, something to which Eugene Wigner paid

[13] *Ibidem*, p. 22.

[14] Cf. A. Gopnik, A.N. Meltzoff, P.K. Kuhl, *The Scientist in the Crib. What Early Learning Tells Us About the Mind*, Harper Paperbacks, New York 2000, chapter 4.

considerable attention.[15] The precise applicability of mathematics in science is explained by the Platonists by reference do the abstract universe of mathematical objects which determines existence of the physical world,[16] just like it is in case of Roger Penrose or the existence of a property, which Michael Heller describes as *the mathematicity of world*.[17]

Does mathematical Platonism satisfactorily explain the internal normativity of formal sciences? Due to the lack of an unambiguous definition of Platonism, the answer to this question is not *explicite* binary, but *implicite* it is reduced to such one. In my opinion the answer is negative. If we assume the strong version of Platonism, that is, one in which the normativity of mathematics is guaranteed by the existence of platonic universe of mathematical truths and objects, and the work of the mathematician is to discover them, the problem of access to this mysterious universe arises. This access is provided by the *intellectual intuition*, postulated by the Platonists, which can be described as a *mathematical insight*. The problem is that this ability is at least as mysterious as the universe of mathematical objects itself, so it is difficult to defend it from the neuroscience point of view. On the plus side, intuition understood as an automatic ability to do make judgments and decisions, is being studied by psychologists.[18] Moreover, mathematicians undoubtedly use many kinds of heuristics but there is a significant difference between them and the insight postulated by Platonists.

Difficulties with mathematical intuition are one of the sources of serious argument against the strong version of mathematical Pla-

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[15] Cf. E.P. Wigner, *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*, "Communications on Pure and Applied Mathematics" 1960, vol. 13, pp. 1–14.
[16] Cf. R. Penrose, *The Large, the Small, and the Human Mind*, The Press Syndicate of the University of Cambridge, Cambridge 1997, chapter 1.
[17] Cf. M. Heller, *Czy Wszechświat jest matematyczny?*, [in:] *idem, Filozofia i Wszechświat*, Universitas, Kraków 2006, pp. 48–57.
[18] Cf. J. Haidt, *The Emotional Dog and Its Rational Tail. Social Intuitionist Approach to Moral Judgment*, "Psychological Review" 2001, vol. 108, no. 4, pp. 814–834.

tonism formulated by Paul Benacerraf.[19] Assuming the theory of knowledge posited by Alvin Goldman,[20] the necessary condition for the existence of the knowledge of a person *X* about the event (or fact) *Y* is the existence of the casual relationship between belief *X* and the event (or fact) *Y*. Benacerraf noticed that the existence of such a causal relationship between objects from the abstractive Platonic universe and the tangible, i.e. embodied, mathematicians, is impossible. Generally speaking, according to Bencerraf, in mathematical Platonism it is impossible to reconcile cognitive abilities and the subject of knowledge. In such an approach, even if the mathematics exists beyond time and the universe, due to a lack of a relationship with those 'living' mathematicians, it is not an excuse for normativity.

Of course, we can try to undermine this argument in different ways.[21] One way is to reject such a strong version of Platonism – which many mathematicians and philosophers of mathematics do, assuming a lighter approach such as the *physicalistic Platonism* presented by Penelope Maddy according to set theory.[22] She eliminates the troublesome intellectual intuition for enhancing the status of ordinary sensory perception. According to Maddy, we can empirically perceive only the elements of finite – and a few – collections, and more complicate collections we treat them like theoretical entities. It is not known for sure whether this approach is still Platonism or rather should be regarded as a form of Aristotelianism, combined with conventionalism. Taxonomic issues are not a major problem either since Maddy's proposition certainly belongs to mathematical realism. More important is that Maddy doesn't deliver an explanation of the normative nature of mathematics. While small collections of sensory per-

[19] Cf. P. Benacerraf, *Mathematical Truth*, [in:] *Philosophy of Mathematics. Selected Readings*, eds. P. Benacerraf, H. Putnam, *op. cit.*, pp. 403–420.
[20] Cf. A. Goldman, *A Causal Theory of Knowing*, "Journal of Philosophy" 1967, no. 64, pp. 357–372.
[21] Cf. M. Balaguer, *Platonism and Anti-Platonism in Mathematics*, *op. cit*., pp. 24–47.
[22] Cf. P. Maddy, *Physicalistic Platonism*, [in:] *Physicalism in Mathematics*, ed. A.D. Irvine, Kluwer, Dordrecht 1990, pp. 259–289.

ceived elements are not a problem, her theory cannot adequately account for normativity related to more complex set-theoretic entities.

The normativity of formalism

If Platonism does not deliver an adequate explanation for the normativity of mathematics, it is worth considering whether this explanation can be derived from a conception from the other side of 'the philosophical barricade', which is – in some ways – formalism. It is commonly assumed that the creator of this approach was David Hilbert. It is hard to regard formalism as a 'philosophical doctrine' as Platonism is. Formalism was rather a research program, annonced in 1900 by Hilbert, whose objective was to strengthen mathematics but we cannot deny that formalism refers to clear philosophical decisions.[23] In the matter of the ontology of mathematics, it is commonly considered that it refers to nominalism while in the matter of epistemology and the conception of mathematical subject, to Kantianism. Hilbert writes:

> Kant taught (…) that mathematics treats a subject matter which is given independently of logic. Mathematics, therefore, can never be grounded solely on logic. Consequently, Frege's and Dedekind's attempts to so ground it were doomed to failure. As a further precondition for using logical deduction and carrying out logical operations, something must be given in conceptualization, viz., certain extralogical concerete objects which are intuited as directly experienced prior to all thinking. For logical deduction to be certain, we must be able to see every aspect of these objects, and their properties, differences, sequences, and contiguities must be given, together with the objects themselves, as something which cannot be reduced to something else

[23] Cf. R. Murawski, *Filozofia matematyki. Zarys dziejów*, PWN, Warszawa 2001, pp. 124–136; 173–203; J.R. Brown, *Philosophy of Mathematics. A Contemporary Introduction to the World of Proofs and Pictures*, 2nd ed., Routledge, New York-London 2008, pp. 67–76.

> and which requires no reduction. This is the basic philosophy which I find necessary, not just for mathematics, but for all scientific thinking, understanding, and communicating. The subject matter of mathematics is, in accordance with this theory, the concrete symbols themselves whose structure is immediately clear and recognizable.[24]

While such an approach raises less controversy in the case of symbols and representing natural numbers, in mathematics there occurs the much more problematic term of *infinity.* Due to the problems involved in defining what is infinity, two strategies are possible. Firstly, we may reject troublesome parts of mathematics (as the intuitionists did, with Brouver at the forefront). But we need to remember that by doing so we throw the baby out with the bathwater – we get rid of the really crucial parts of mathematics. The second option is to look for the method which will let us strengthen mathematics along with the problematic term of infinity.

In the writing of his famous sentence "No one shall drive us out of the paradise which Cantor has created for us",[25] Hilbert has chosen the second option. Yet, he was convinced that only the application of finite methods would guarantee certainty in mathematics. He made the distinction between finitistic mathematics, whose sentences refer only to objects which have specific content (these sentences can be defined as real sentences) and infinitistic mathematics, whose sentences are Kant's ideas of pure reason, but they are not implemented in the real world (we can define these as ideal sentences). It is extremely important, Hilbert said, that for any finite mathematical sentence it is possible to provide finitistic proof. On the other hand, the appeal to infinitistic methods may perform only an auxiliary role in mathematics, e.g. in the construction of elegant proofs. As a rule, any infinistic proof may be – according to Hilbert – changed into a finit-

[24] Cf. D. Hilbert, *On the Infinite*, [in:] *Philosophy of Mathematics. Selected Readings*, eds. P. Benacerraf, H. Putnam, *op. cit.*, p. 192.
[25] *Ibidem*, p. 191.

istic proof. To this purpose he created the proof theory, which is a branch of metamathematics.

The program of formalism, also known as a Hilbert program, may be divided into two, seperate parts. The first part was about the reconstructing of mathematics as a formalized system. This involved the construction of an artifical symbolic language, setting the rules of building proper formulas in it and the choice of axioms and rules of inference. The essence of formalism lies in the fact that evidence had strictly syntactic features that mean, they were related to the shape of the signs only, not to their meaning. In such an approach, the taking of proof has algorithmic, in other words, strictly machine features (compatible with a Turing machine). An important feature of the formal system is that it should be consistent. It should be understood by the fact that. on the basis of an axiomatic system, an evidence of any correctly constructed (in terms of system language) sentence, or negation of this sentence, can be presented. In other words, if the sentence *A* is a correctly constructed sentence (in the language of that system), it can be proved that either sentence *A* or sentence *~A.* The second part of the Hilbert program was to prove the consistency and conservativeness of infinitistic mathematics. Consistency means that in the system no pair of contradictory sentences exists. On the other hand, by proving the conservativeness, the idea was to show that any sentence that has an infinitistic proof can also be proved by means of finitistic methods. Hilbert and his students – mainly Ackermann – managed to achieve some of the program's objectives. An important, and indeed final blow to Hilbert's program was given in 1930, when Kurt Gödel (as we remember – the supporter of mathematical Platonism) proved his famous *first incompleteness theorem*[26] and announced the *second incompleteness theorem*.

[26] "Any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete. In particular, for any consistent, effectively generated formal theory that proves certain basic arithmetic truths, there is an arithmetical statement that is true, but not provable in the theory", S.C. Kleene, *Mathematical Logic*, Wiley, New York 1967, p. 250.

There is a commonly held perception that Hilbert's program transformed mathematics into a game of meaningless symbols. But we need to remember that, in the original version, the formalistic view of mathematics was a methodology, not ontology.[27] Although the discussed problem is still an object of active dispute, it seems that, contrary to the commonly held perception, from Hilbert's view it is impossible to completely eliminate the semantic element (which is an otologically important issue). Moreover, as Platonists would probably say – we also can't eliminate the intuitive element (epistemology). In order to explain this issue I will cite the reconstruction of Hilbert's program, as proposed by Bartosz Brożek and Adam Olszewski:

> (STAGE1) Identification of the beyond any doubts, the finitistic part of the real mathematics (with content)
> (STAGE2) The formalization of this part of mathematics (but it has no difference whether this formal part will be termed real mathematics or already ideal mathematics)
> (STAGE3) The construction of the relevant formal system (axioms and inference rules), from which it is possible to reconstruct the finistic part of the mathematics. Formalized semantic mathematics in this process a heuristic function, it can be said, that on this stage axioms are treated as if they are semantic.
> (STAGE4) Treatment of the constructed formal system as a set of finite signs.
> (STAGE5) Proof of consistency (and if need be other mathematical proofs) of the formal system. Those proofs are proved in metamathematics, which is finitistic and semantic (we operates on finite symbols)
> (STAGE6) Mathematical (semantic) evaluation ("interpretation") of the theorems generated by the formal system. The consistency, which

[27] However, it should be noted, that there exist versions of formalism with strong ontological commitments, see H.B. Curry, *Outlines of a Formalist Philosophy of Mathematics*, North-Holland, Amsterdam 1951.

> was proven before, of the system guarantees the truth of outcome theorems (…).[28]

Assuming, for the purposes of this paper, a certain idealization, it can be said that the practice of the formalism is three-tier:

- ***First level***: 'encoding' of relevant part of the mathematics, setting axioms and inference rules. The nature of this level is *semantic*.
- ***Second level***: when conducting mechanic computations and proving evidences under an axiom system. All symbols are treated as *without content* (*formal*). On this level, in order to authenticate the results of computations, consistency, completeness and decidability of the formal system are checked. The nature of this level is *syntactic* (*formal*).
- ***Third level***: The results of computation are subject of *interpretation*. The nature of this level is *semantic*.

The formalized mathematics is treated as a game of meaningless symbols o*nly on the second level*. It should be emphasized once more that formalism is a methodological and heuristic resolution but it is not ontological.

Returning to the normativity of mathematics, the formal approach doesn't solve this problem for at least two reasons. Firstly, considering the remarks formulated above, formalism as a whole contains semantic elements. Therefore, most of the problems in which the mathematical Platonism is involved return. Above all, it is yet unknown what is the nature of rules which guide the mathematician during operations mentioned on the first and the third level. A paradoxical sounding statement can be risked in that nothing stands in the way of a declared Platonist (in the strong meaning of this word) to protect the thesis about the contact of a 'formalistic'

[28] B. Brożek, A. Olszewski, *Podmiot matematyczny Hilberta* (draft).

mathematician with the 'Platonic universe of mathematical ideas'. But then we come back to the problem of intellectual intuition (insight). Yet this is not all.

Formalism does not formulate a satisfactory solution to normativity also in the case of operations on meaningless symbols which are conducted on the *second level.* As I mentioned before, on the second level computations are mechanical. More precisely, mathematician proving theorems under the axiom system behave like the Turing Machine. In this context, Wittgenstein's argument against a mechanistic interpretation of rule-following can be cited. In *A Wittgenstein Dictionary* Hans-Johann Glock writes, that in mechanism: "Understanding a rule is a disposition, and statements about dispositions are ultimately statements about a mechanism".[29] In a mechanistic approach every rule, e.g. "+2" is the same as (identical) with the disposition to answer a specific question. The Turing Machine runs in a *deterministic way* – the actual state of machine determines its future states. Following the rule (e.g. "+2") is *completely automatic*. In such an approach, it is impossible to act against the rule. Of course, we may try to identify actions against the rule with the machine mistake but, based on the definition, a Turing Machine doesn't make mistakes and doesn't break down. Furthermore, it means that a cause–effect, approach of rule-following is far from what we commonly understand as normativity.[30]

In view of my previous analysis, two representative philosophies of mathematics (if it may be said so about formalism) are unable to cope with the explanation of the phenomena of the normativity of mathematics. If two representative conceptions concerning philosophy and the foundations of mathematic do not deliver a satisfactory solution of the problem that we are interested in, it needs to be referred to other hypotheses.

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[29] H.J. Glock, *A Wittgenstein Dictionary*, Blackwell, MA-Oxford 1996, p. 325.

[30] Cf. B. Brożek, R. Zyzik*, Wittgenstein o regułach*, "Logos i Ethos" 2008, vol. 24, no. 1, pp. 27–57.

3. Where Does the Normativity of Mathematics Come From?

Anti-foundationalism

In his famous essay with the meaningful title of "Against Foundationalism" Michael Heller describes the position which is the object of his attack:

> Philosophers very often (not to say: notoriously) share two great ambitions: firstly; to build the philosophical system, which will be based on irrefutable grounds and secondly; that this system will be an irrefutable grounds for other branches for knowledge. Both of those ambitions, although essentially distinct, usually go together.[31]

Both Platonism and formalism may certainly be considered as foundationalist positions. This is because they refer to immutable and universal rules (strong Platonism) and they claim to deliver the foundations for mathematics (formalism). In my opinion, much better approach to explaining the riddle of normativity of mathematics is anti-foundationalism. In such an approach, philosophical conceptions should be treated not as *dogmas*, but analogical to scientific *hypotheses* and *theories*, which are vulnerable to *revision*. As Michael Heller writes:

> Obviously, we can deal without formulating of the initial hypotheses (...). But they should be the hypotheses, not the "irrefutable" or "obvious" axioms. These hypotheses should be formulated on the basis of the current knowledge from particular branch and on the base of knowledge about history of particular problems, but (…) they always

[31] Cf. M. Heller, *Przeciw fundacjonizmowi*, [in:] *idem*, *Filozofia i Wszechświat*, Universitas, Kraków 2006, p. 83.

> will be assisted by some kind of visionary element. We do not need to pretend that it does not exist, we should try to control it instead (…). From the accepted initial hypotheses we derive our conclusions (…). But it is worth to go a step further and to derive some kind of feedback between our initial hypothesis and the conclusions, that were derived from them (…). A proper built system says something about its hypotheses. Due to this process the initial hypotheses become enhanced (become less "hypothetic"), that leads to "enhancing" of conclusions deducted from them. Multiple repetition of this process may give us something close to certainty.[32]

It is worth adding that hypotheses formulated by philosophers should take into account the *best* and the *most current* knowledge. Nowadays, perhaps, most developed branches of knowledge are the sciences about the brain and mind: neuroscience and cognitive science. I believe that referring to them may lead us to interesting outcomes also in the matter of the riddle of the normativity of mathematics.

Proto-normativity and full-blooded-normativity

To shed better light on the normativity of mathematics it is necessary to make some distinctions. By adapting the theory of normativity constructed by Bartosz Brożek,[33] I will distinguish two types of rules: *proto-rules* and *full-blooded-rules* bounded-up with *proto-normativity* and *full-blooded-normativity*.[34] One of the justifications of this distinction is the hypothesis that it corresponds both to the phylogenetic and ontogenetic development of the disposition of Homo sapiens to follow rules.

[32] *Ibidem*, pp. 96–97.
[33] Cf. B. Brożek, *Normatywność prawa*, Wolter Kluwers, Kraków 2012, *passim*.
[34] Bartosz Brożek (cf. *ibidem*) talks about *rudimentary-normativity* (*rudimentary-rules*) and *abstract-nornativity* (*abstract-rules*).

Now let us define, along with Bartosz Brożek and Radosław Zyzik, what are those proto-rules. They say, that proto-rules "(…) are some kind of emergent creations of mental states and social behaviors – or they supervenes on them".[35] This conception is one of the possible interpretations of rules in the Wittgenstein approach. In the following part of this paper I will try to explain this definition using tools derived from the embodied-embedded mind paradigm. At the current stage I would like to distinguish two essential features of proto-rules. The first of them is evolutionary – both phylogenetic and ontogenetic – anteriority in language. The second is the 'nonlinear entanglement' of different types of rules. By this I understand that on the level of proto-normativity it *is impossible to distinguish* rules which we will describe as moral, linguistic or mathematical.

When it comes to full-blooded-rules (and full-blooded-normativity) they are involved in the language practices of Homo sapiens. What more can be said is that those rules are strictly dependent on language. On this level the specific types of rules – due to language and social behaviors – become *individualized*. By this I mean that we are able distinguish moral, mathematical, linguistic (and other) rules. The reason for rules also becomes individualized. Moral and ethical rules are reasons for human actions (in this context we most often call them duties); linguistic rules are reasons for the proper use of language, while mathematical and logical rules are reasonsfor accuracy, stability, intersubiectivity or the necessity in the formal sciences. So, the normativity of mathematics is specific in nature. Although, as I said, full-blooded-rules are vulnerable to individualization, it doesn't mean that they cannot influence each other or even overlap.

Both the proto-rules and the full-blooded-rules satisfy two conditions. The first may be described as *condition of reason* – the rule may be the reason for action. The second may be described as *condi-*

[35] B. Brożek, R. Zyzik, *Wittgenstein o regułach*, *op. cit.*, p. 30.

tion of pattern – the rule is the pattern of proceeding (ethical, linguistic or mathematic).[36] Thanks to these conditions, the breaking of the rule may be connected with relevant sanction.

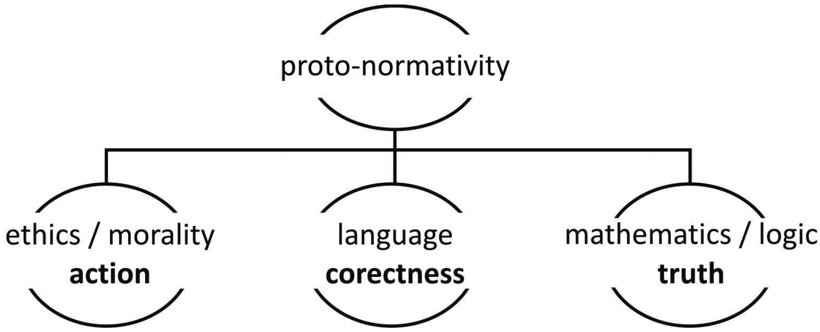

Figure show the development of proto-normativity into the full-blooded-normativity of particular domains

Proto-rules and embodied mind

As we remember, proto – rules "(…) are some kind of emergent creations of mental states and social behaviors – or they supervenes on them".[37] Because in this definition the emphasis falls on the mind and social behaviors, in my opinion, an appropriate instrument to highlight the problem of normativity is second generation cognitive science, defined as embodied-embedded mind. While the first generation cognitive science, popular in the 1980s, treated mind as a computer program (software) installed on biological hardware, the embodied-embedded paradigm mind is a result of the *interaction* between the body (that is why we call it *embodied*) and its environment. It is also said that mind is *embedded* in a physical, social and cultural environment. Since the embod-

[36] Cf. B. Brożek, *Normatywność prawa*, *op. cit.*, *passim*.
[37] B. Brożek, R. Zyzik, *Wittgenstein o regułach*, *op. cit.*, p. 30.

ied mind was constructed in the 1980s in the context of linguistic works of George Lakoff and Ronald Langcaker, it nowadays has a biological basis. Lakoff and Vittorio Gallese, in the paper they wrote together, describe the assumptions of the embodied mind paradigm in the following way:

1. ***Information structure***. (…) The information structure needed to characterize the conceptual structure of grasp is available at the neural level in the sensory-motor system (…).
2. ***Multimodality***. Mirror neurons and other classes of premotor and parietal neurons are inherently "multimodal" in that they respond to more than one modality. Thus, the firing of a single neuron may correlate with both seeing and performing grasping. Such multimodality (…), meets the condition that an action-concept must fit both the performance and perception of the action.
3. ***Functional clusters***. Multimodality is realized in the brain through functional clusters, that is, among others, parallel parietal-premotor networks. These functional clusters form high-level units – characterising the discreteness, high-level structure, and internal relational structure required by concepts.
4. ***Simulation***. To understand the meaning of the concept grasp, one must at least be able to imagine oneself or someone else grasping an object. Imagination is mental simulation (…), carried out by the same functional clusters used in acting and perceiving (…).
5. ***Parameters***. All actions, perceptions, and simulations make use of neural parameters and their values. For example (…) the action of grasping an object makes use of the neural parameter of force (…). The same parameter values that characterise the internal structure of actions and simulations of actions also characterize the internal structure of action concepts.

> 6. ***Structured neural computation***. The neural theory of language (…) provides a theory of neural computation in which the same neural structures that allow for movement and perception in real time and in real sensory-motor contexts also permit real-time context-based inferences in reasoning (…).[38]

As we can see, in the embodied mind a key role is played by the *sensorimotor system*. One of the justifications of such an approach was the discovery of motor-control programs by Srini Narayanan.[39] During the modeling of the motor system he discovered that all of its structures have the same control structure, which include: 'getting into a state of readiness', 'the initial state', 'the starting process', 'the main process (either instantaneous or prolonged)', 'an option to stop', 'an option to resume', 'an option to iterate or continue the main process', 'a check to see if a goal has been met', 'the finishing process', 'the final state'. Every physical activeness conducted by our body, e.g. drinking from a cup, owes a stable structure and is using above schemes. Moreover, it shows that the same schemes are functioning also in the language and describe a conceptual structure of events. Linguists call it an *aspect*[40] and, if it's true, our cognitive abilities are indeed embodied. As evidenced by George Lakoff and Rafael Núñez, the motor-control program plays a key role in case of our mathematical abilities.[41] Let us first take a look at basic mathematical abilities.

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[38] V. Gallese, G. Lakoff, *The Brain's Concepts: the Role of the Sensory-Motor System in Conceptual Knowledge*, "Cognitive Neuropsychology" 2005, vol. 21, pp. 3–4.

[39] Cf. S. Narayanan, *Embodiment in Language Understanding. Sensory-Motor Representations for Metaphoric Reasoning About Event Descriptions*, Ph.D. Dissertation, Department of Computer Science, University of California, Berkeley 1997; *idem*, *Talking the Talk Is Like Walking the Walk. A Computational Model of Verbal Aspect*, [in:] *Proceedings of the Nineteenth Annual Conference of the Cognitive Science Society*, eds. M.G. Shafto, P. Langley, Erlbaum, New Jersey 1997.

[40] Cf. G. Lakoff, M. Johnson, *Philosophy in the Flesh. The Embodied Mind and its Challenge to Western Thought*, Basic Books, New York 1999, p. 41.

[41] Cf. G. Lakoff, R. Núñez, *Where Mathematics Comes From. How the Embodied Mind Brings Mathematics into Being*, Basic Books, New York 2000, pp. 34–37.

From proto-mathematics to embodied mathematics

For a long time it has been known that some mathematical abilities have an inborn nature or show themselves in the early months of life. Moreover, some of those abilities are shared with non-human primates. In this context, developmental psychologists observed the following facts. Within three or four days after birth, children are able to visually distinguish collections of two and three elements. A little later this ability extends to four element collections. At four months, children have proto-arithmetical abilities. They understand that 1 + 1 = 2 and 2 – 1 = 1. A little later they are able to see that 1 + 2 = 3 and 3 – 1 = 2. As we can see, children understand that those operations are commutative. At about seventh months of life, children begin to notice the equivalence between the amount of visual and auditory stimuluses.[42] There is a need to add that, at the age of four, children learn to add bigger numbers using their fingers. Neural corellates of the simple numeric abilities are located in the structures of the inferior parietal cortex, with particular emphasis on the agular gyrus, which is located in the Brodmann area 39. Basic mathematical abilities can be defined, after Bartosz Brożek, as *embrained*.[43] Generally, it can be said that little children are equipped with elementary 'intuition of sets' and 'the number sense.'[44] But there is need to remember that those abilities are still very limited. The huge problem which developmental psychologists, cognitive scientists, and philosophers of mathematics face is the 'crossing of Rubicon', which separates the above mentioned proto-mathematical abilities from mathematics in its best – this means all mathematical theories – those we learn about in school and those which professional mathematicians use.

[42] Cf. *ibidem*, pp. 15–16.
[43] Cf. Brożek, *Neuroscience and Mathematics: From Inborn Skills to Cantor's Paradise*, [in:] *Between Philosophy and Science*, eds. M. Heller, B. Brożek, Ł. Kurek, Copernicus Center Press, Kraków 2013.
[44] Cf. S. Dehaene, *The Number Sense. How the Mind Created Mathematics*, Revisted and Expanded Edition Oxford University Press, Oxford-New York 2011.

One of the propositions for the 'crossing of the Rubicon' is illustrated by Elizabeth Spelke. She claims that the transition to more complicated mathematical abilities occurs due to language. According to Spelke, by absorbing language children learn to connect numeric with geometric representations, leading to the creation of abstract terms which are necessary in mathematics.[45]

Contrary to the data emphasised by Platonists, language – or, more generally, human conceptual systems – plays a very important role in human mathematical abilities. One of the most sophisticated proposals for explaining the phenomena of mathematics is the embodied cognitive science of mathematics proposed by George Lakoff and Rafael Núñez in the book *Where Mathematics Comes From*.[46] In their opinion, mathematical abilities aren't separated from other cognitive abilities of humans. On the contrary, they are strictly connected with our ordinary experiences and interactions in which our bodies meet with the environment. In particular, they emphasise the role of spatial relation concepts such as: up – down, inside – outside, or image-schemas, such as: 'the above schema', 'the contact schema', 'the support schema' or 'the source-path-goal schema.'[47] An essential role for mathematical abilities is also played, as mentioned above, Narayanan's motor-control programs. However, the basic instrument used by Lakoff and Núñez is the *cognitive theory of metaphor*. Metaphor is not understood as a poetic mean (as in Aristotelian tradition), but as an 'instrument of thinking and acting':

> Conceptual metaphor is a cognitive mechanism for allowing us to reason about one kind as if it were another. This means that metaphor is not simply a linguistic phenomenon, a mere figure of speech. Rather it is a cognitive mechanism that belongs to the realm of thought (…).

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45 Cf. E. Spelke, *Natural Number and Natural Geometry*, [in:] *Space, Time and Number in the Brain. Searching for the Foundations of Mathematical Thought*, eds. S. Dehaene, E. Brannon, Elsevier, London 2011, pp. 287–317.

46 Cf. G. Lakoff, R. Núñez, *Where Mathematics Comes From*, *op. cit*.

47 Cf. *ibidem*, pp. 27–49.

> “Conceptual metaphor” has a technical meaning: It is a *grounded, inference-preserving cross-domain mapping* – a neural mechanism that allows us to use the inferential structure of one conceptual domain (say, geometry) to reason about another (say, arithmetic). Such conceptual metaphors allow us to apply what we know one branch of mathematics in order to reason about another branch.[48]

Metaphorization, understood as a cognitive ability, consists of transferring the significant structures from the physical world into the domain of concepts. Thanks to metaphors, it is possible to create abstract concepts such as mathematical concepts.[49] According to Lakoff and Núñez, arithmetic is created through the dispositions described above, such as the number sense, and conceptual metaphors, which includes: ‘Arithmetic As Object Collection’, ‘Arithmetic As Object Construction’, ‘the Measuring Stick Metaphor’ or ‘Arithmetic As Motion Along a Path’. For example ‘Arithmetic Is Object Collection’ works as following (on the left side of arrow is the *Source Domain* (*Object Collection*) and on the right *Target Domain* (*Arithmetic*)):[50]

- Collection of objects of the same size → Numbers
- The size of the collection → The size of the number
- Bigger → Greater
- Smaller → Less
- The smallest collection → The unit (One)
- Putting collections together → Addition
- Taking a smaller collection from a larger collection → Subtraction

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48 *Ibidem*, p. 6.

49 Cf. *ibidem*, *passim*; W.P. Grygiel, M.L. Hohol, R. Piechowicz, *Zmetaforyzowana matematyka i zmatematyzowana metafora*, “Logos i Ethos” 2011, vol. 31, no. 2, pp. 147–168; M.L. Hohol, *Matematyczność ucieleśniona*, [in:] *Oblicza racjonalności. Wokół myśli Michała Hellera*, eds. B. Brożek, J. Mączka, W.P Grygiel, M.L. Hohol, Copernicus Center Press, Kraków 2011, pp. 143–166.

50 Cf. G. Lakoff, R. Núñez, *Where Mathematics Comes From*, *op. cit.*, p. 55.

Under the embodied cognitive science of mathematics, Lakoff and Núñez, apart from arithmetic, also explain such branches as Boolean algebra ('Boole's Metaphor'), set-theory ('Classes are Containers'), or mathematical analysis (e.g. 'Dedekind's Metaphors', 'Weierstrass's Continuity Metaphor'). Moreover, they propose explanations of even such concepts as *infinity* ('The Basic Metaphor of Infinity', 'Cantor's Metaphor') or the *formal reduction* ('The Formal Reduction Metaphor'). As a case study, which authenticates their research, they conduct detailed cognitive analysis of the famous Euler equation $e^{\pi i}+1=0$.

The program of the embodied cognitive science of mathematics described briefly above is undoubtedly an interesting theory. Despite the preliminary nature and some inaccuracies, the theory of Lakoff and Núñez is, perhaps, the only proposal of the systematic explanation of mathematics which refers to human cognitive abilities. The theory of metaphors constructed by Lakoff and his co-workers is indeed the theory of concepts. To a lesser extent, it talks about the language practices that operate within society. In my opinion, although reference to the proposal described in *Where Mathematics Comes From* is very valuable, it doesn't completely solve the problem of the normativity of mathematics, because it only concerns the origins of proto-rules and proto-normativity. In order to understand the adoption to rules an full-blooded-normativity, it needs to be explain exactly how proto-rules and proto-normativity are enhanced by social practices and, therefore, how mathematics can enjoy the resulting stability. In my view, it is possible with reference to the second part of the embodied-embedded mind paradigm, i.e. by embedding the mind in the social and linguistic practices of Homo sapiens.

Embedded mind: culture through imitation

In his book *The Cultural Origins of Human Cognition*, Michael Tomasello reflects upon the riddle of the rapid evolution of Homo sapiens, from sophisticated tools and symbols, advanced communication techniques to complex public institutions. In his opinion, the 6 million years that separate the evolution of humans from the evolution of other humanoids is definitely too short a period of time for natural selection to have made such spectacular achievements. Those achievements cannot be explained either by genetic specifity – because the genetic similarity between humans and chimpanzees is 99%. Tomasello looks for the uniqueness of human creations in the one feature which distinguishes humans from other humanoids:

> My particular claim is that in the cognitive realm the biological inheritance of humans is very much like that of other primates. There is just one major difference, and that is the fact that human beings "identify" with con specifics more deeply than do other primate.[51]

Based on the studies from such disciplines as evolutionary biology, anthropology, primatology and developmental psychology, he formulated a hypothesis according to which in the progress of biological evolution, humans have created a *unique form of social cognition*, which has enabled *cumulative cultural evolution*. He calls this process 'the ratchet effect'. Tomasello writes:

> The process of cumulative cultural evolution requires not only creative invention but also, and just as importantly, faithful social transmission that can work as a ratchet to prevent slippage backward-so that the newly invented artifact or practice preserves its new and improved form at least somewhat faithfully until a further modification or improvement comes along. Perhaps surprisingly, for many animal species it is not the creative component, but rather the stabilizing ratchet component, that is the difficult feat. Thus, many nonhuman primate individuals regularly produce intelligent behavioral innovations and novelties, but then their group mates do not engage in the

[51] M. Tomasello, *The Cultural Origins of Human Cognition*, Harvard University Press, Cambridge 1999, p. 14.

> kinds of social learning that would enable, over time, the cultural ratchet to do its work (…).[52]

To be more precise, the Tomasello hypothesis has three dimensions. In the first, phylogenetic dimension, he emphasizes that Homo sapiens have achieved an unparalleled ability to identify with other individuals of their own species, which has led to the perception of other people as aware, intentional and endowed with minds. This unusual ability, not found in other species, is possible due to the ability of imitation. In the second, historical dimension, the 'cultural ratchet' has led to the creation of new forms of learning, means of communication and unique cultural creations. In the third, ontogenetic dimension, Tomasello emphasizes that children are born and grow up in a formed and stable cultural and social reality. Thanks to this, they don't need to rediscover all of their cultural creations.[53]

The research conducted by Tomasello and his co-workers at the Max Planck Institute for Evolutionary Anthropology in Leipzig and several field observations shows that, among non-human primates, *intentional teaching* doesn't exist. Although it was observed that among non-human primates that *emulation* is frequent – a situation when an observer is interested not in the behavior itself, i.e. not in the exact sequence of movements but in its purpose, or changes in the enviroment caused by this behavior. However, *imitation* wasn't observed. Imitation is when the attention of the observer is focused on the behaviour itself, on a fixed sequence of movements. This hypothesis may seem to be counter-intuitive until we realize that, because of the details of behaviour, strict imitation require the involvement of many more cognitive resources than emulation. According to Tomasello, the specifically human ability to imitate promotes the process of intentionally and purposeful teaching, which results in the creation of the world of culture. This is why Tomasello speaks

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[52] *Ibidem*, p. 5.
[53] *Ibidem*, p. 10.

of people as ‘imitation machines’. The role of imitation in creating a culture is appreciated by other scientists also, such as Merlin Donald.[54]

The key role in the process of imitation is played by the mirror neurons system. In Giacomo Rizolatti’s view, mirror neurons may resonate in two ways, namely: recognition of purpose (high level resonance) and recognition of method (low level resonance). He claims that only Homo sapiens can fully use those both ways, which results in a ‘combinatoric explosion’. It is based on the fact that humans are able to use the same method for the realization of different purposes and to realise the same purpose by different methods.[55] Research conducted by Marco Iacobni shows that the frontal area is responsible for purpose recognition and the parietal area of the mirror neuron system is responsible for the recognition of methods. Systems of mirror neurons play a key role in the process of the simulation of the mental states of other people.[56] Thanks to mirror neurons, people aren’t ‘solipsists’, who are certain about only their own mental states, but they are able to read the feelings or intentions of other people. This ability is called the *Theory of Mind*. In the ontogenetic view – according to Tomasello –two breakthroughs in the life of a child are important for the formation of the Theory of Mind. Until nine months, children do not differ from other primates in the case of cognitive abilities. The first breakthrough comes exactly in the ninth month, when children learn to perceive other people as intentional beings.[57] It is not surprising then that this breakthrough is called *The Nine-Month Revolution*. In this breakthrough, a very important role is played by the

[54] Cf. M. Donald, *Imitation and Mimesis*, [in:] *Perspectives on Imitation vol. 2: Imitation, Human Development, and Culture*, eds. S. Hurley, N. Chater, MIT Press, Cambridge, Mass. 2005, pp. 283–300.
[55] Cf. G. Rizolatti, *The Mirror Neuron System and Imitation*, [in:] *Perspectives on Imitation vol. 1: Mechanisms of Imitation and Imitation in Animals*, eds. S. Hurley, N. Chater, MIT Press, Cambridge-Mass. 2005, pp. 55–76.
[56] Cf. S. Baron-Cohen, *Mindblindness*, The MIT Press, Boston 1997, pp. 31–58.
[57] Cf. M. Tomasello, *The Cultural Origins of Human Cognition, op. cit.*, pp. 61ff.

ability to imitate. The second breakthrough comes at about the age of four, when among children the Theory of Mind is fully developed. Then children start to treat other people as owners of minds and learn to predict their mental states.

What is crucial from the riddle of normativity point of view is that treating other people as intentional beings starts before the mastery of language. According to Tomasello, gaining abilities such as: creating of analogies, metaphorization or taking a perspective of other people is earlier than language. Language – which is above all the system of effective communication and whose important part is stability – must be embedded on the biologically based stability of actions. This conception was called a 'Culture First' conception by Merlin Donald, who is convinced that language was formed on the basis of gesture communication. The 'Culture First' conception is contrary to the 'Language First' conception. As I have already mentioned, Theory of Mind allows us to identify with other members of our species and is very important for the creation of social intersubjectivity. Moreover, Tomasello is convinced that this cognitive ability also made possible the creation of scientific ideas, such as, for example, the relationship of cause and effect:

> (…) Human beings built directly on the uniquely primate cognitive adaptation for understanding external relational categories, they just added a small but important twist in terms of mediating forces such as causes and intentions. (…) Moreover, my hypothesis is that, just as primate understanding of relational categories evolved first in the social domain to comprehend third-party social relationships, human causal understanding also evolved first in the social domain to comprehend others as intentional agents.[58]

The Theory of Mind and the ability to imitate plays – according to Tomasello – a significant role for the sociogenesis of mathematical abilities:

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[58] *Ibidem*, pp. 23–24.

> The history of mathematics is an area of study in which detailed examination has revealed myriad complex ways in which individuals, and groups of individuals, take what is passed on to them by previous generations and then make modifications as needed to deal with new practical and scientific problems more efficiently (…). My hypothesis (…) is that building on the basic primate sense of quantity, human beings also use their formidable skills of perspective-taking and alternative construals of concrete objects and collections of objects (which have a social basis in skills of perspective-taking and linguistic communication) to construct complex mathematics.[59]

From the proto-normativity to the full-blooded-normativity of mathematics

According to the conception I have described above, the 'cultural ratchet' is the factor that sustains the stability of such domains as language or mathematics. I think that, due to imitation and based on the human-specific form of social cognition, in the course of social interactions, the rules of full-blooded-normativity emerges. This conception also applies – in my opinion –to mathematics. To the creation of the normativity of mathematics it is not enough the similar biological construction of the mathematicians and, based on it sense of number and the intuitiveness of plural, neither the strict similarity of conceptual structures in embodied minds of the mathematicians. Those conditions are *necessary* but *not sufficient*. Only the stable social interactions among mathematicians causes the emergence of the normativity of mathematics. Thanks to this, it is possible to follow common rules which provides the ability to say if somebody has made a mistake: "you have done this incorrectly, check it". What is important is that the conception described above supports the idea of distinguishing between *proto-rules (proto-normativity)*, which are earlier than language and the mathematical *full-blooded-rules (full-blooded-*

[59] *Ibidem*, p. 46.

normativity), which emerge and are enhanced on the basis of the language practices of Homo sapiens.

To sum up, we can distinguish three basic stages of this theory, which explains where mathematics and its normativity comes from:

- ***Embrained mathematics***: inborn or gained at the early childhood proto-mathematical abilities, such as the intuitiveness of sets or the number sense (Daheane); at this stage we can only talk about proto-normativity only.
- ***Embodied mathematics***: the conceptual system based on the functioning of the sensorimotor system; an essential part of the human conceptual system are mathematical metaphors (Lakoff and Núñez), on this stage proto-normativity still occurs.
- ***Embedded mathematics***: the 'cultural ratchet', based on the specific human ability to imitate, which provides the ability to follow rules (Tomasello, Donald); only at this stage does the full-blooded normativity of mathematics emerge.

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The conception of the normativity of mathematics presented above is of a highly speculative nature yet I do not see anything wrong in the speculative nature of philosophical conceptions if they meet criteria of *disputability*. According to Michael Heller, this criterion is the equivalent of Popper's falsifiability in the empirical sciences. I am convinced that some elements of the conception I proposed are falsifiable (in the Popper sense), but the whole conception is disputable (in the sense of Heller).

The three most controversial matters which need to be discussed are: (1) the transition from *embrained* to *embodied-embedded mathematics*, (2) the nature of Lakoff and Núñez' embodied cognitive science of mathematics proposal and (3) the problem of the applicability of mathematics in sciences. So far as embodied and embedded mathematics are strictly connected to each other, we may see the gap between them and embrained mathematics. The proposal of Elizabeth Spelke concerning the linguistic nature of the transition from proto-mathematics to more advanced mathematical abilities is compatible with my conception, but it raises a lot of controversy among developmental psychologists and cognitive scientists. In the case of Lakoff's and Núñez' proposal, it raises ambivalent feelings. On the one hand, it is the only conception which tries to explain where the *whole* of mathematics comes from in a systematic way (without omitting even such terms as infinity). On the other hand, many decisions made by the authors are of an aprioristic nature.

Finally, on the basis of my conception, there remains the riddle of the matter of the applicability of mathematics in science. In my opinion, we can approach this problem in two ways. The first is afforded by interpretation: the concepts (including mathematical concepts), that are shaped by our embodied minds in some way *fit* reality. Based on this principle, scientists are sometimes able to adjust mathematical theories to the empirical world. It is an approach represented by Lakoff and his co-workers.[60] The second interpretation is, in my opinion, closer to the conception of the *mathematicity of the world* by Michael Heller.[61] If, on the basic level, proto-rules do not have a linguistic nature, and mathematical concepts arise due to the interaction between the organism and the real world, nothing stands in the way of saying that embrained-embodied-embedded mathematics, is mathematics spelt with a lowercase 'm', resonating with the greater whole which is mathematics spelt with a capital 'M'.

[60] Cf. G. Lakoff, R. Núñez, *Where Mathematics Comes From*, *op. cit*., pp. 337–382; G. Lakoff, M. Johnson, *Philosophy in the Flesh*, *op. cit*., pp. 74–129.
[61] M. Heller, *Czy Wszechświat jest matematyczny?*, *op. cit.*